Missing, Delayed, or Muddled Topics in Common Core's Math Standards

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Kindergarten -- Grade 7: In mathematics standards, the words used to define the level to which students are to learn mathematical procedures or skills are *proficiency, mastery,* and *automaticity*, in that order However, *automaticity* and *mastery* never appear in Common Core's mathematics standards (CCMS). In fact, *proficiency* appears with one exception only in the chapter *Standards for Mathematical Practice*, and only in the phrase "mathematical proficiency" or "mathematically proficient student."

The only word used in CCMS that could be interpreted as meaning one of those three words is "fluency." But Common Core embeds its meaning in the phrase "procedural fluency" (defined as "skill in carrying out procedures flexibly, accurately and appropriately") without explaining the kind of procedures students would carry out in this manner or the level of skill they should reach. Nor does it use the phrase more than once: it almost always uses "fluency with" or "fluently ... using" as a substitute.

Thus we find no requirement in CCMS that students reach the level of automaticity:

- for addition and subtraction with the standard algorithms or any other algorithms,
- for multiplication with the standard algorithm or any other algorithm, and
- for division with the standard long division algorithm or any other division algorithm.

Not only is there no requirement for automaticity with the standard algorithms of arithmetic, there is no requirement for automaticity for any core skill or procedure mentioned in CCMS. Moreover, mathematical errors occur in more advanced material, together with very careless writing.

Ratios and Proportional Relationships

Grade 6: Understand ratio concepts and use ratio reasoning to solve problems.

1. Understand the concept of a ratio and use ratio language to describe a ratio relationship between two quantities. *For example, "The ratio of wings to beaks in the bird house at the zoo was 2:1, because for every 2 wings there was 1 beak." "For every vote candidate A received, candidate C received nearly three votes.*"

> **Comment:** "nearly" does not correspond to ratio or rate in any way. At best it corresponds to a range of ratios, but the tools for handling such objects are not covered until college and require advanced calculus.

3. Use ratio and rate reasoning to solve real-world and mathematical problems, e.g., by reasoning about tables of equivalent ratios, tape diagrams, double number line diagrams, or equations.

b. Solve unit rate problems including those involving unit pricing and constant speed. *For example, if it took 7 hours to mow 4 lawns, then at that rate, how many lawns could be mowed in 35 hours? At what rate were lawns being mowed?*

> **Comment**: There is no indication of the size of the lawns or the amount of time it takes to mow each. Rather, the *assumption* is that they all take the same time to mow. Suppose some were 5000 square feet and some were 8000 square feet. We do not know the amount of time it takes to mow 8000 square feet compared to 5000 or if some lawns were steeply sloped and others level.

Grade 7: Analyze proportional relationships and use them to solve real-world and mathematical problems.

2. Recognize and represent proportional relationships between quantities.

    a. Decide whether two quantities are in a proportional relationship, e.g., by testing for equivalent ratios in a table or graphing on a coordinate plane and observing whether the graph is a straight line through the origin.

    Comment: High-achieving countries first define that *two points in the coordinate plane, (a,b) and (c,d), are in a proportional relationship if and only if neither is (0,0) and they both lie on a single straight line through the origin.* Presuming that a is non-zero, then writing $b = ra$ (so r = b/a), we see that d must equal rc for the two points to form a proportion. Without this starting point, the following sub-standards are confusing.

    b. Identify the constant of proportionality (unit rate) in tables, graphs, equations, diagrams, and verbal descriptions of proportional relationships.

    c. Represent proportional relationships by equations. *For example, if total cost t is proportional to the number n of items purchased at a constant price p, the relationship between the total cost and the number of items can be expressed as t = pn.*

    d. Explain what a point (x, y) on the graph of a proportional relationship means in terms of the situation, with special attention to the points (0, 0) and (1, r) where r is the unit rate.

    Comment: What is the graph of a proportional relationship? Any straight line through (0, 0)?

3. Use proportional relationships to solve multi-step ratio and percent problems. *Examples: simple interest, tax, markups and markdowns, gratuities and commissions, fees, percent increase and decrease, percent error*.

    Comment: The given examples are all relatively trivial; the only non-trivial example (compound interest) is not included here or elsewhere in CCMS. Also, it is not clear why "percent error" occurs as an application example as late as grade 7 when the formula for solving it (see below) is also given:

$$\frac{\text{(measured value)} - \text{(exact value)}}{\text{exact value}} \times 100$$

    High- achieving countries introduce ratios and rates in grade 3 or 4 and students are expected to have mastered rate problems by grade 5 or 6.

We also find that:

- CC fails to teach decimals until grade 4, about two years behind high-achieving countries.
- CC fails to teach key geometrical concepts usually taught in K-7 (e.g., sum of angles in a triangle, isosceles and equilateral triangles) until the high school geometry standards.
- CC excludes conversion between different forms of fractions: regular fractions, decimals, and percents. The word "conversion" appears in four CCMS standards (noted below), but all four are undemanding and the grade 7 standard has a minor mathematical error:

    * The grades 4 and 5 standards (4.MD.1 and 5.MD.1) ask for conversion within a single system of units, e.g. feet to inches and centimeters to meters.

    * Standard RP.3d asks students to "Use ratio reasoning to convert measurements units."

    * 7.NS.2d asks students to "convert a rational number to a decimal using long division." Then it

asks students to "know that the decimal form of a rational number terminates in 0's or eventually repeats."

> **Comment:** The phrase "the decimal form of a rational number terminates in 0's or eventually repeats" is redundant since *eventually repeats* includes terminating in 0. More important, this phrase should be a separate standard and not look like an afterthought. The key point of 7.NS.2d should have been that not every real number is the decimal expansion of a fraction. In other words, real numbers contain rational numbers.
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> Another problem with 7.NS.2d is that there is no FINITE algorithm for adding or multiplying real numbers, so the standard lacks meaning. The process of "conversion" should have been discussed more carefully.

- CC fails to teach prime factorization. Consequently, it does not teach least common denominators or greatest common factors, although "least common multiple" and "greatest common factor" are mentioned in Standard 6.NS.4 (with a puzzling example whose point is not stated):

  > Find the greatest common factor of two whole numbers less than or equal to 100 and the least common multiple of two whole numbers less than or equal to 12. Use the distributive property to express a sum of two whole numbers 1-100 with a common factor as a multiple of a sum of two whole numbers with no common factor. For example, express 36 + 8 as 4 (9 + 2).

  Since prime factorization is not discussed in CCMS, general methods for determining least common denominator or greatest common factor are not available to students. All they can reasonably be asked to do is to (laboriously) work out specific examples.

- CC omits teaching of compound interest and the formula for calculating it:

$$\frac{x^{n+1} - 1}{x - 1} = 1 + x + x^2 + \cdots + x^n.$$

  This is a grade 7 or 8 topic in high-achieving countries, and was a grade 7 topic in the old California standards. CCMS provides a standard for this kind of problem in high school (Standard A-SSE.4), but too late and without sufficient background information and explanation:

  > Derive the formula for the sum of a finite geometric series (when the common ratio is not 1), and use the formula to solve problems. *For example, calculate mortgage payments.*

Summary: The meaning of "rigorous"

The academic level of the CCMS is dramatically lower than the level in high-achieving countries even though school administrators, self-described policy makers, as well as education school faculty generally describe CCMS as "rigorous." So, how did this word get to be used to describe CCMS?

In some cases, people using this term have had as their subjective reference the pre-CC standards in the weakest states, e.g., the 2003 Missouri or Wyoming mathematics standards. Many have said that CCMS are more rigorous than the state standards they replaced. But there may be a better explanation for the use of this term to describe CCMS.

The *Dictionary of Education Reform* (edglossary.org/rigor) offers the following definition of "rigor":

> "While dictionaries define the term as rigid, inflexible, or unyielding, educators frequently apply (the terms) rigor or rigorous to assignments that encourage students to think critically, creatively, and more flexibly. Likewise, they may use the term rigorous to describe learning environments that are not intended to be harsh, rigid, or overly prescriptive, but that are stimulating, engaging, and supportive."

Consequently, when ordinary parents hear such people using the word "rigorous" to describe CCMS, they are likely to misunderstand what the speaker actually means. It is not possible to describe CCMS as extremely thorough, careful, or even accurate, since many of the standards are undemanding, omit many key topics, are unclearly written, and even have mathematical errors. At best, education professionals who use "rigorous" to describe CCMS may well be saying that its standards promote "creativity and flexible thinking" (although they do not indicate how) and they may also be implying absolutely nothing about accuracy or intellectual demand.

On the other hand, for foundational mathematics, neither creativity nor flexibility is desired. We do not want students to decide what 37/7 *should* or *could* be. Rather, we want them to be able to say that 37/7 is 5 + 2/7, or 5(2\over 7), or perhaps 5.\overline (285714).

Algebra 1: Missing components needed for Algebra 1 or beyond

1. Division of monomials and polynomials with remainder. Indeed, there is only one mention of polynomial remainders. It occurs on page 64 and refers to the simplest possible case, "the remainder theorem" which determines the remainder on dividing by x - a.
2. Derivation and understanding of the properties of slopes of parallel and perpendicular lines (that could been done easily using CC's formulation of geometry in Euclidean transformational terms.

$$\frac{(ax+b)}{(cx+d)(ex+f)} = \frac{r}{(cx+d)} + \frac{s}{(ex+f)}$$

3. Manipulation and simplification of rational expressions. In particular, the basic property (partial fraction decomposition) for arbitrary (a, b), with its key applications to graphing and understanding rational functions, as well as basic preparation for pre-calculus, calculus, and more importantly for the solutions of differential equations in engineering and the sciences.[1]
4. Multi-step problems with linear equations and inequalities
5. Multi-step problems using all four operations with polynomials
6. Multi-step problems involving manipulation of rational expressions
7. Solving two (or more) linear inequalities in two variables and sketching the solution sets
8. Basic addition and half angle formulas for sin and cosine.
9. Any preparation for limits.
10. Almost no development of the standard properties of ellipses, hyperbolas, and parabolas, such as the existence and properties of the foci and directrix.

Geometry: Some missing key topics

Properties of triangles and circles: Students should know that:

- All three perpendicular bisectors of a triangle always intersect at a single point.
- Every triangle is circumscribed by a unique circle with a center at the intersection point of the three perpendicular bisectors of the edges.
- Every right triangle has the center of the circumscribing circle on its hypotenuse.
- The angle subtended by an arc on the circle (the angle obtained by drawing the two lines from the center to the ends of the arc) is twice the angle subtended by the ends of the arc and any point on the circle which is in the interior of the complement of the arc.

Issues with CCMS geometry. The geometry standards are very prescriptive, explaining exactly how they want the subject to be taught. The chosen method is non-standard and not validated by research. Indeed, some 35 years ago, the method was adopted in the former Soviet Union for the most advanced students but rapidly abandoned because it simply didn't work. CCMS requires geometry to be based on the properties of "the Euclidean Group" - the set of transformations of the coordinate plane consisting of reflections about any straight line, rotations through any angle with center at any point in the plane as well as these rotations and reflections

1. William McCallum, a lead writer of the CCMS, co-authored a college textbook *Harvard Calculus* that also omitted partial fraction decomposition.

followed by a translation.

For example, a key grade 8 geometry standard (Standard 8.G.2) is incomprehensible as written, to students and teachers:

> Understand that a two-dimensional figure is congruent to another if the second can be obtained from the first by a sequence of rotations, reflections, and translations; given two congruent figures, describe a sequence that exhibits the congruence between them.

Teachers have never seen anything like it, and students will wonder, "I've always heard that two figures are congruent if they have the same shape and size. How does 8.G.2 relate to this?" To add further confusion to the story, on page 64 of CCMS we find:

> "For triangles, congruence means the equality of all corresponding pairs of sides and all corresponding pairs of angles. During the middle grades, through experiences drawing triangles from given conditions, students notice ways to specify enough measures in a triangle to ensure that all triangles drawn with those measures are congruent. Once these triangle congruence criteria (ASA, SAS, and SSS) are established using rigid motions, they can be used to prove theorems about triangles, quadrilaterals, and other geometric figures."

Teachers now have to discuss "measures," that, somehow, congruence doesn't change, and that we have to somehow show that the equality of these measures ensures that two triangles with these measures are congruent. This is so advanced in reasoning and logic that only an unusual student in K-12 will have some idea of what this means. And in Standard 8.G.4, we find something worse:

> "Understand that a two-dimensional figure is similar to another if the second can be obtained from the first by a sequence of rotations, reflections, translations, and dilations; given two similar two-dimensional figures, describe a sequence that exhibits the similarity between them."

Up to this point both students and teachers have understood that two figures are similar "if they have the same shapes but not, necessarily, the same size." What do *dilations* have to do with this? We don't know and never find out. Nowhere in CCMS are dilations defined. Thus it is not surprising that even top students in the countries of the USSR, among the highest achieving countries in the world in mathematics, were unable to handle this approach in K-12.

Algebra II: Some key missing topics

1. Writing quadratic polynomials in two or three variables as sums or differences of perfect squares. (KEY for the study of conic sections, which, in turn, underlies a massive amount of the preliminary material in all STEM areas.)
2. Detailed study of surfaces of revolution coming from quadratic polynomials as described above. In particular, the focus here should be on parabolic mirrors and their applications.
3. Introduction of the foci and the directrix for conics and their applications to parabolas and parabolic mirrors, as well as for ellipses and elliptic surfaces with applications to things like whispering galleries and Kepler's laws.
4. Definition and implications of the eccentricity for conic sections.
5. Structure of logarithms to base 10, e, or general base, $b > 0$. Conversion between bases, calculation of explicit values in simple cases.

Algebra II: Missing components needed for Calculus

- Composite functions (for example functions of the form $f(g(x))$ if the domain of f contains the range of g). There is one and only one mention of composite functions (F-BF.4b) and then only in the context of one of the most special cases possible.
- Combinations and permutations. (There is only one mention of them, and only in a (+) standard on page

82 of CCMS, S.CP.9. But they and associated binomial coefficients form the basis for virtually all combinatorial results that are used in many, if not most, real-world applications of mathematics.)

- Finite and infinite arithmetic and geometric sequences
- Mathematical induction

All four topics above are quite ``formal'' in line with the overly formal treatment of algebra in Common Core's Standards. But they are much more ``realistic'' in terms of the actual needs of students wishing to major in any technical area in college.

Pre-calculus and/or Algebra II, Trigonometry: Some key missing topics

1. Partial fraction decomposition of relatively simple rational functions and their graphs. The partial fraction decomposition obtained in (Algebra 1, Missing components, item 3) has r and s determined as the solutions of the two linear equations in two unknowns, $er + cs = a$, and $fr + ds = b$. One can always find r and s as long as $cx + d$ is not a constant multiple of $ex + f$, since this implies that the determinant $cf - ed \neq 0$. So this system of linear equations has one and only one solution.

   This is one of the key applications of the systems of linear equations that are supposed to be studied in Algebra I or earlier, to say nothing of the addition formula for fractions.) The other key application, *linear regression* (determining the regression line of a set of data points in the plane) is far too advanced for high school mathematics since it requires multi-variable calculus.

2. Graph functions in polar coordinates. Key examples (all standard topics in high-achieving countries):
   Circles written in the form $r = 2\cos(t)$,
   Cardioids ($2 + 2\cos(t) = r$),
   Rose petal curves ($r = \sin(5t)$), and
   Lemniscates ($r^2 = 4\sin(2t)$).

Issues with the definition of vectors in CCMS. All standards relating to vectors in CCMS are (+) topics, but they are confusing to the point that they contain mathematical errors. For example, consider Standard N-VM.1:

1. (+) Recognize vector quantities as having both magnitude and direction. Represent vector quantities by directed line segments, and use appropriate symbols for vectors and their magnitudes (e.g.,v, |v|, ||v||, v).

> **Comment:** This standard confuses vectors (points in the coordinate plane that one adds coordinate-wise [$(A, B) + (a,b) = (A+a, B+b)$] and multiplies by a number (scales) via the rule $c(A, B) = (cA, cB)$ -) with the field of (tangent) vectors on the plane, and this confusion continues in the remaining standards in this section, as the next example illustrates.

2. (+) Solve problems involving velocity and other quantities that can be represented by vectors.

> **Comment:** Velocity means "velocity AT A POINT," which associates a tangent vector at the point in question to the point. In other words, velocity involves the field of tangent vectors, not the elements in a single vector space. What could it possibly mean to add velocities at *two different points?*